Purpose

Measure a wheel’s angular velocity and angular acceleration which are caused by the torque of the wheel. Through the measurement, we can verify Newton's laws of motion and other physical quantities of a rotating rigid body around a fixed axis. The main measuring method is to rotate a wheel by external torques and then measure the rotating wheel’s angular velocity and angular acceleration and other rotating physical quantities.

Theory

When an object rotates around point O, the angular position at t_1 and t_2 are θ_1 and θ_2 so the average angular velocity of the object is

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} = \frac{\text{Angular displacement per unit}}{\text{Per unit time}}$$

The angular velocity can be written as

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(rad/s or rev/s)}$$

Similarly, assume the angular velocity of t_1 and t_2 are ω_1 and ω_2 so the average angular acceleration is

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

Therefore, the angular acceleration can be written as

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad (\text{rad}/s^2 \quad \text{or} \quad \text{rev}/s^2)$$

Since the kinetic energy is the collection of all rigid bodies, we can then know

$$K = \frac{1}{2}m_1 {v_1}^2 + \frac{1}{2}m_2 {v_2}^2 + \frac{1}{2}m_3 {v_3}^2 + ... = \sum_i \frac{1}{2}m_i {v_i}^2$$

Substitute $v = r\omega$ into the above equation:

$$K = \sum_i \frac{1}{2}m_i (r_i\omega)^2 = \frac{1}{2}(\sum m_i {r_i}^2)\omega^2 = \frac{1}{2}I\omega^2$$

The moment of inertia I can be defined as

$$I = \sum_i m_i {r_i}^2$$

The unit of I is $kg \cdot m^2$. If the particle is a continuous body,

$$I = \int r^2 dm = \int \rho r^2 dV \qquad (\rho \text{ is the density})$$

From the torque

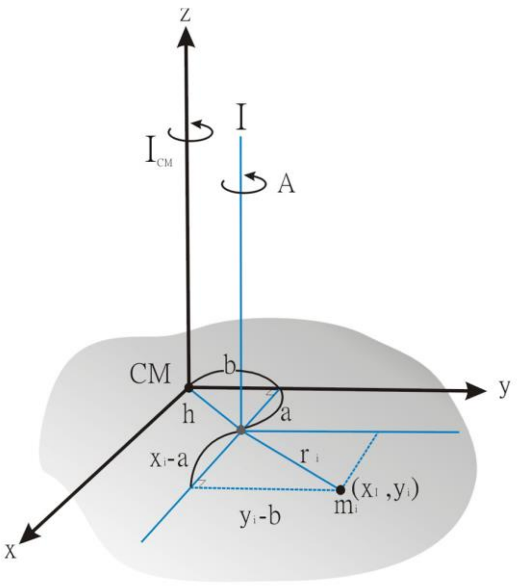
$$\tau = F_t r = m(\alpha r)r = (mr^2)\omega = I\alpha$$

Note: parallel axis theorem

If the total mass of an object is M, the moment of inertia which passes through the axis of centroid is I_{CM} , the moment of inertia of another parallel axis is I and the distance between two axes is h, the formula can then be expressed as:

$$\underline{I = I_{CM} + Mh^2}$$

Proof:



Assume the rotation axis direction is axis Z and the centroid is the origin of the coordinate ($x_{CM} = y_{CM} = 0$) so the moment of inertia of particle m_i to axis A is

$$I_i = m_i r_i^2 = m_i[(x_i - a)^2 + (y_i - b)^2] = m_i(x_i^2 + y_i^2) + (a^2 + b^2)m_i - 2am_i x_i - 2bm_i y_i$$

$$\therefore I = \sum I_i = \sum m_i(x_i^2 + y_i^2) + (a^2 + b^2)\sum m_i - 2a\sum m_i x_i - 2b\sum m_i y_i$$

The final two terms are zero because the centroid defines

$$\sum m_i x_i = Mx_{CM} = 0$$
$$\sum m_i y_i = My_{CM} = 0$$

And $a^2 + b^2 = h^2$

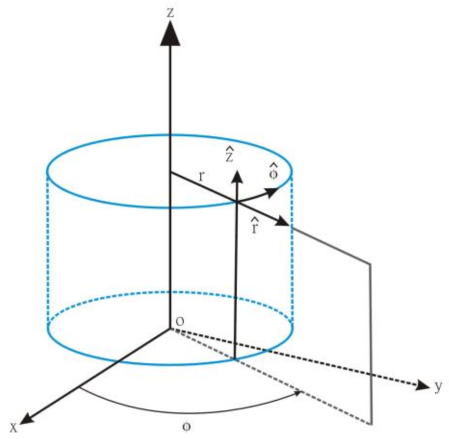
$$\therefore I = \sum m_i(x_i^2 + y_i^2) + (\sum m_i)h^2 = I_{CM} + Mh^2$$ Q.E.D

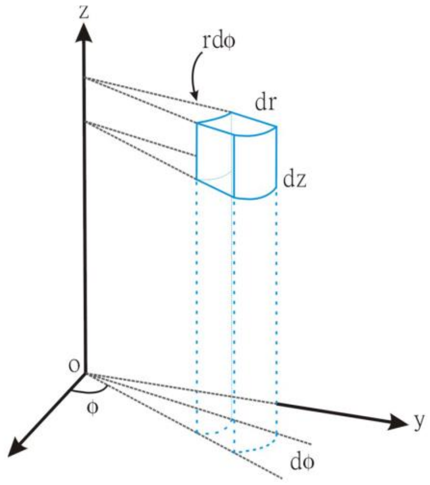
The derivation of the moment of inertia I:

When calculating the moment of inertia, we need to first know the relationship of the coordinate. Their relationship can be expressed as follows.

Cylindrical coordinates:

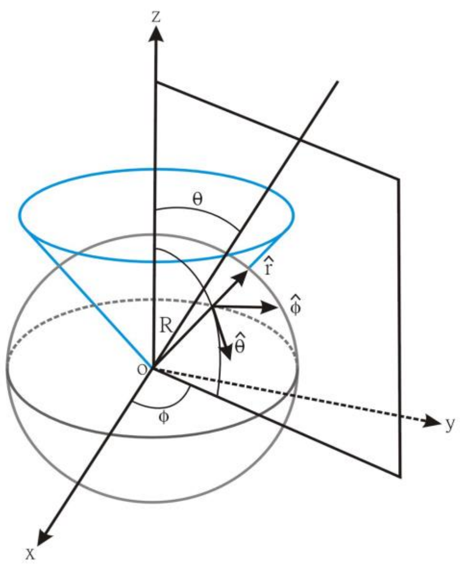
$$x = r\cos\phi \qquad y = r\sin\phi \qquad z = z \qquad dv = r\,dr\,d\phi\,dz$$

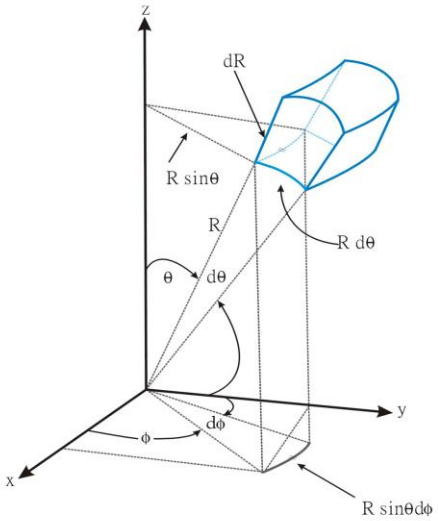




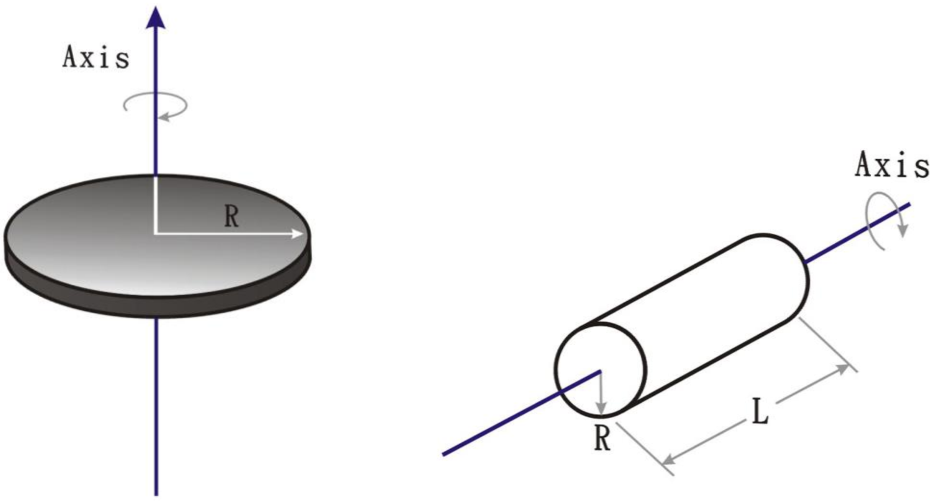
Spherical coordinates:

$$x = R\sin\theta\cos\phi \qquad y = R\sin\theta\sin\phi \qquad z = R\cos\theta \qquad dv = R^2\sin\theta\,dR\,d\phi\,d\theta$$





1. A disk or a cylinder: I= $\frac{1}{2}MR^2$



Assume the above two figures are the very small volume of cylinders so

$$dv = r dr d\phi dz$$

Assume the body density is ρ so dm is

$$\text{dm} = \rho dv = \rho r dr d\phi dz$$

So

$$I = \int r^2 dm = \int r^2 \rho dv = \rho \int_0^R r^3 dr \int_0^{2\pi} d\phi \int_0^l dz = \rho \frac{R^4}{4}(2\pi)l = \frac{1}{2}\rho\pi l R^4$$

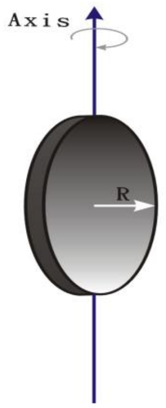
Because

$$\rho = \frac{M}{\pi R^2 l}$$

Hence,

$$I = \frac{1}{2}\rho\pi l R^4 = \frac{1}{2}\left(\frac{M}{\pi R^2 l}\right)\pi l R^4$$
$$= \frac{1}{2}MR^2$$

Whether the object is a disk or a cylinder, *I* is not influenced by the height of the object.

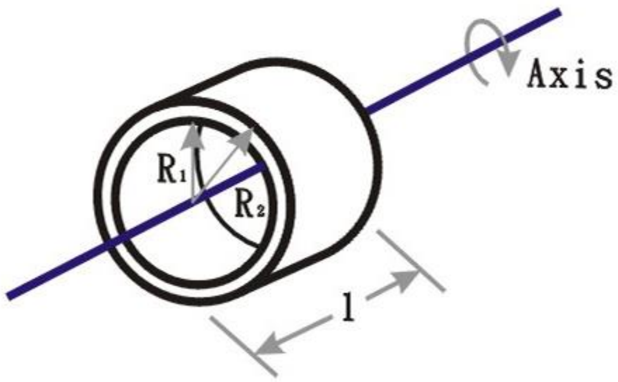


Also, assume we have an object like above figure and we change the position of the rotation axis so the moment of inertia of the disk is

$$I = \int r^2 dm = \int (r\sin\phi)^2 \rho r dr d\phi dz = \rho \int_0^R r^3 dr^{3} \int_0^{2\pi} \sin^2\phi d\phi \int_{-\frac{d}{2}}^{\frac{d}{2}} dz = \frac{M}{\pi R^2 d} \cdot \frac{R^4}{4} \cdot \pi \cdot d$$

$$= \frac{1}{4} MR^2$$

2. An annular cylinder: I= $\frac{1}{2} M (R_1^{\ 2} + R_2^{\ 2})$



Similar to the above equation, if we change the upper and lower limitation of r in the integral equation,

$$I = \int r^2 dm = \rho \int_{R_1}^{R_2} r^3 dr \int_0^{2\pi} d\phi \int_0^l dz = \frac{1}{2} \rho \pi d (R_2^{\ 4} - R_1^{\ 4})$$

and

$$\rho = \frac{M}{\pi (R_2^{\ 2} - R_1^{\ 2}) l}$$

$$\therefore I = \frac{1}{2}\frac{M}{\pi(R_2^{\,2} - R_1^{\,2})l}\pi l(R_2^{\,4} - R_1^{\,4})$$

and

$$\frac{(R_2^{\,4} - R_1^{\,4})}{(R_2^{\,2} - R_1^{\,2})} = \frac{(R_2^{\,2} + R_1^{\,2})(R_2^{\,2} - R_1^{\,2})}{(R_2^{\,2} - R_1^{\,2})}$$

$$\Rightarrow I = \frac{1}{2}M(R_1^{\,2} + R_2^{\,2})$$

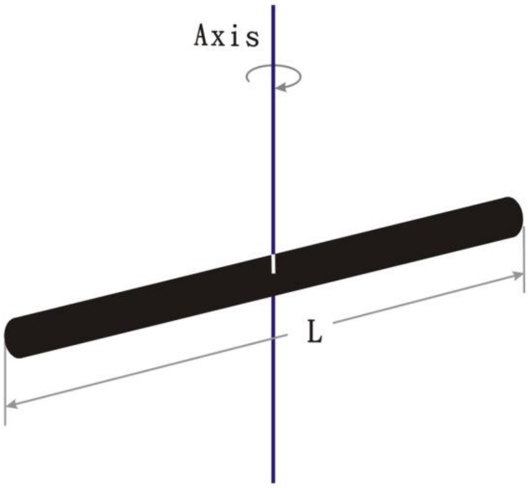
3. A cylinder

(i) Assume the cylinder radius can be <u>neglect</u> and the density is uniform. If the column density is $\lambda = \frac{M}{L}$ and the stick is at position x, the formula is

$$dm = \lambda dx = \frac{M}{L}dx$$

$$I = \int x^2 dm = \int x^2 \lambda dx$$

The result will be different because of different centre rotation positions (or the rotation axis positions)

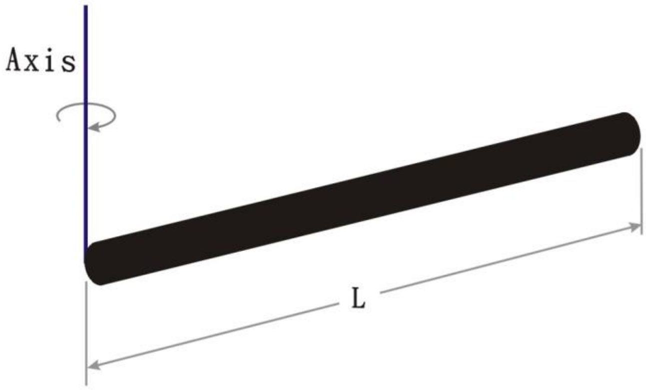


If the axis is at the stick centre (shown as the above figure),

$$I = \int x^2 dm = \int x^2 \lambda dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \lambda dx = 2\int_0^{\frac{L}{2}} x^2 \lambda dx = 2\frac{M}{L}\frac{(L/2)^3}{3}$$

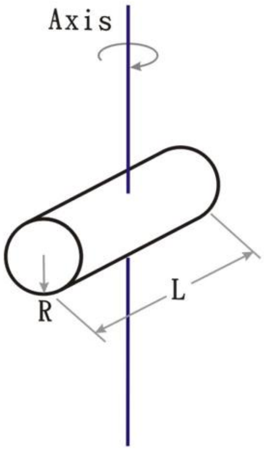
$$= \frac{1}{12}ML^2$$

If the axis is at the side end of the stick,



$$I=\int x^2 dm=\int x^2\lambda dx=\int_0^L x^2\lambda dx=\frac{1}{3}L^3(\frac{M}{L})$$
$$=\frac{1}{3}ML^2$$

(ii) Assume the cylinder radius can not be neglect and its density is uniform. Here we only discuss the situation of horizontal rotation.



We slice the cylinder into small disks and the mass of each disk is dm. Assume the rotation axis is the diameter of the disk and its moment of inertia which we have calculated is $I=\frac{1}{4}MR^2$. From the parallel axis theorem, we know

$$I=\int dI=\int d(I_{CM}+Mh^2)=\int dI_{CM}+\int dm\cdot dh^2$$

From the above figure, we know $I = I_{CM} = \frac{1}{4}MR^2 . \int dm \cdot dh^2$ in the parallel axis is equal to the calculation of the cylinder (neglect the radius) at the axis centre which we have derived. So

$$I = \int dI_{CM} + \int dm \cdot dh^2 = \int d(\frac{1}{4}MR^2) + \int x^2 dm = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

R is the cylinder radius and L is the length of the cylinder.

4. Spherical body (solid and hollow)

(i) Solid sphere